METHOD OF THE LEAST SQUARED DISTANCES

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1. Introduction

One of the most important tasks in 3-dimensional (3D) object location is the solving of a system of linear equations:

$$\begin{cases} \left(c_{i9} \times \overline{X}_{ai} - c_{i1}\right) \times X_a + \left(c_{i10} \times \overline{X}_{ai} - c_{i2}\right) \times Y_a + \left(c_{i11} \times \overline{X}_{ai} - c_{i3}\right) \times Z_a = c_{i4} - \overline{X}_i \\ \left(c_{i9} \times \overline{Y}_{ai} - c_{i5}\right) \times X_a + \left(c_{i10} \times \overline{Y}_{ai} - c_{i6}\right) \times Y_a + \left(c_{i11} \times \overline{Y}_{ai} - c_{i7}\right) \times Z_a = c_{i8} - \overline{Y}_i \\ i = \overline{1, N} \end{cases} \tag{1}$$

Where:

$A\left[X_a, Y_a, Z_a\right]$ – 3D coordinates of the point A is in the base coordinate system $OXYZ$

$\overline{A}_i\left[\overline{X}_{ai}, \overline{Y}_{ai}\right]$ – Object's 2D projection coordinates is in the coordinate system $\overline{O}_i\overline{X}_i\overline{Y}_i\overline{Z}_i$ connected to the sensor #**i**

c_{ij} – Calibration coefficient

N – Total number of sensors

The goal is to find 3D coordinates of the point $A\left[X_a, Y_a, Z_a\right]$ from the measured object's 2D projection coordinates on N sensors.

Equation (1) can be covered by the following general system:

$$\begin{cases} d_{i0} + \sum_{j=1}^{M} \left(d_{ij} \times X_i\right) = 0 \\ i = \overline{1, N} \end{cases} \tag{2}$$

Where:

$A\left[X_1, X_2, ..., X_M\right]$ – 3D coordinates of the point A is in the base coordinate system

d_{ij} – Coefficient

N – Total number of sensors

M – Number of unknowns

2. Solution

The goal of this method is to find the sum of the least squared distances from an M-dimensional point to each M-dimensional plane, which is the solution of the system (2). Each equation of system (2) represents a plane in M-dimensional space. If no distortions are present, then M-dimensional planes intersect at one M-dimensional point. Otherwise the number of intersection points is more than one. The distance from point $A\left[X_1, X_2, ..., X_M\right]$ to each M-dimensional plane, determined by measurement number i, is the following:

$$\begin{cases} \delta_i = \dfrac{d_{i0} + \sum\limits_{j=1}^{M}\left(d_{ij} \times X_i\right)}{\sqrt{\sum\limits_{j=1}^{M} d_{ij}^{2}}} \\ i = \overline{1, N} \end{cases} \tag{3}$$

The sum of all squared distances is:

$$\sum_{i=1}^{N} \delta_i^{2} = \sum_{i=1}^{N}\left(\frac{d_{i0} + \sum\limits_{j=1}^{M}\left(d_{ij} \times X_i\right)}{\sqrt{\sum\limits_{j=1}^{M} d_{ij}^{2}}}\right)^{2} = \sum_{i=1}^{N}\left(c_{i0} + \sum_{j=1}^{M}\left(c_{ij} \times X_i\right)\right)^{2} = Min \tag{4}$$

The partial derivatives of (4) generate the following system of linear equations:

$$\frac{\partial\left(\sum\limits_{i=1}^{N} \delta_i^{2}\right)}{\partial\left(X_i\right)} = 2 \times \sum_{i=1}^{N}\left(\frac{\left(d_{i0} + \sum\limits_{j=1}^{M}\left(d_{ij}^{2} \times X_i\right)\right)}{\sqrt{\sum\limits_{j=1}^{M} d_{ij}^{2}}}\right)$$

The minimum of sum (4) can only be when all partial derivatives are equal to zero, leading to the following system of ***M*** equations with ***M*** unknowns:

$$\begin{cases} 2 \times \left(\sum\limits_{i=1}^{N}\left(c_{i0} + \sum\limits_{j=1}^{M}\left(c_{ij} \times X_i\right)\right)\right) \times \sum\limits_{j=1}^{M} c_{ij} = 0 \\ i = \overline{1, N} \end{cases} \quad \text{or} \quad \begin{cases} \sum\limits_{i=1}^{N}\left(c_{i0} + \sum\limits_{j=1}^{M}\left(c_{ij} \times X_i\right)\right) = 0 \\ i = \overline{1, N} \end{cases} \tag{5}$$

Where:

$$c_{i0} = \frac{d_{i0}}{\sqrt{\sum\limits_{j=1}^{M} d_{ij}^{2}}} \qquad c_{ij} = \frac{\sum\limits_{i=1}^{N} d_{i0}}{\sqrt{\sum\limits_{j=1}^{M} d_{ij}^{2}}}$$

The system of equation (5) can be solved by any common method.

3. Summary

The solution derived by the method of "***the least squared sum of distances***" provides improved results for large coefficient spreads, which, unlike common methods, is less dependant and limited by computer resolution.