

Time-Domain Finite Element Method in Electromagnetics

A Brief Review

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Time domain computation of Maxwell equations is required in electromagnetic radiation, scattering and propagation problems. In addition, it is often more economical to get frequency domain results via time domain data than the other way around if many frequencies are involved. The computation in differential form, i.e., the time domain finite difference method was first exploited by Yee in 1966. While the time domain finite element method in electromagnetics developed much later (1984). The finite difference method has shortages in dealing with curved boundaries, inhomogeneous materials, the Neumann boundary conditions, and so on. All of these situations might be encountered, say, in the area of integrated optics which is rapidly growing and posing numerical challenges. The finite element method, being originated from structural mechanics and successful in aircraft design, is hopeful to overcome some the shortages in the finite difference method.

In this article, the algorithm of time domain finite element method is reviewed with focus on semi-discrete Galerkin's weighted residual approach. Its applications to electromagnetics are then discussed. The article consists of four parts. Part A introduces basic ideas of Galerkin's weighted residual method in spatial domains; Part B describes Galerkin's weighted residual method and suppression of Neumann boundary conditions in both time-domain and spatial domains; Part C deals with vector wave equations and discusses stability and convergence; and finally Part D summarizes the article.

Part A. Basic Ideas of Galerkin's Weighted Residual Method

Finite element method is one of many numerical techniques of solving partial differential equations, say, in region Ω :

$$Lu = f \quad (\text{eq. 1})$$

where L is a differential operator with respect to space and time. Boundary conditions and initial conditions may also be specified. The exact solution of (eq.1) can certainly render $Lu - f = 0$. An approximate solution \hat{u} will yield a finite residual R over the region Ω , i.e.,

$$L\hat{u} - f = R \quad (\text{eq. 2})$$

To ensure a best fit of \hat{u} to u , one requires that $R \cong 0$ in Ω . Imagining we have a vector space V , each function is a vector defined in Ω . The residual function is one of the vectors. R will be zero if its inner products with all other vectors in the space V vanish. Here the inner product is defined as

$$(v_1, v_2) = \int_{\Omega} v_1 v_2 d\Omega$$

So that $R = 0$ if

$$(v, R) = 0, v \in V$$

In fact, it is sufficient to test if (w_α, R) vanishes in order to determine $R = 0$ or not. Where w_α with $\alpha = 1, 2, \dots, N$, are the basis functions of space V . This property has a counterpart in the three-dimensional vector space. Consider an arbitrary vector $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$, $\hat{x}, \hat{y}, \hat{z}$ are unit vectors in x, y and z directions respectively. If \vec{A} is zero, then all its components must vanish, i.e., $A_x = 0, A_y = 0, A_z = 0$. On the other hand, since $\hat{x}, \hat{y}, \hat{z}$ consists in a complete basis set in the three-dimensional vector space, $\vec{A} = 0$ is equivalent to the fact that the scalar products between \vec{A} and all the basis vectors are zero, i.e.,

$$\begin{aligned}\hat{x} \cdot \vec{A} &= A_x = 0 \\ \hat{y} \cdot \vec{A} &= A_y = 0 \\ \hat{z} \cdot \vec{A} &= A_z = 0\end{aligned}$$

These steps however cannot be applied directly to space V since its dimension is usually infinity, i.e., $N = \infty$. It is therefore impossible to test for all w_α in practice. Fortunately, it is not necessary to do so since we only seek approximate solution of (eq.1). We will be happy if $R \approx 0$. So that we confine ourselves to a subspace of V which has finite dimension, M . We impose

$$(w_i, R) = 0, \quad i = 1, 2, \dots, M \quad (\text{eq.3})$$

where w_i are the basis functions of the subspace. Since w_i do not consist in a complete basis set of space V , (eq.3) cannot guarantee $R = 0$ everywhere in Ω . However, as long as w_i are properly chosen and M is adequately large, then $R \approx 0$ can be satisfied to any required precision. In other words, $L\hat{u} - f \approx 0$. Here \hat{u} is an approximate solution of (eq.1). The accuracy is certainly dependent on M and on how to select w_i .

Inserting (eq.2) into (eq.3), one leads to

$$\begin{aligned}(w_i, L\hat{u}) &= (w_i, f), \\ i &= 1, 2, \dots, M\end{aligned} \quad (\text{eq.4})$$

This equation, (eq.4), is called the weak form of (eq.1), in mathematical terminology. To obtain \hat{u} , one expand it in terms of trial functions N_i

$$\hat{u} = N_0(\vec{x}) + \sum_{i=1}^M a_i N_i(\vec{x}), \quad i = 1, 2, \dots, M \quad (\text{eq.5})$$

where N_0 is chosen to satisfy the given boundary conditions of the problem.

$N_i (i = 1, 2, \dots, M)$ are termed coordinate functions (or the finite element basis functions) and must each satisfy the corresponding homogeneous form of the boundary conditions. The coordinate

functions must be linearly independent, this means none of them is obtainable as a linear combination of the remainder.

The expansion coefficients $a_i (i = 1, 2, \dots, M)$ are unknowns and to be determined from (eq.4).

Equations (eq.2) to (eq.5) are the essence of the weighted residual method of solving (eq.1).

To implement the above formulation, one has to specify N_i by the following procedures:

(a) Dividing Ω into sub-domains or elements.

(b) Introducing local coordinates and shape functions such that in each element e :

$\hat{u}^e(\bar{x}) = \sum_n \hat{u}_n S_n(\bar{x})$, where n runs over all nodes of element e , \hat{u}_n are nodal values of \hat{u} , and

$S_n(\bar{x})$ are shape functions that may be linear, quadratic or higher order polynomials and have the property that $S_n(\bar{x}) = 1$ if \bar{x} at node n , and $S_n(\bar{x}) = 0$ if \bar{x} at all nodes other than n .

(c) Constructing \hat{u} from \hat{u}^e . \hat{u} is piece-wisely defined by \hat{u}^e . Comparison of \hat{u} thus constructed with (eq.5) indicates that i represents the global node index and the nodal values \hat{u}_i are unknowns in place of a_i . The relationship between N_i and S_n can be specified at this stage. N_i are also defined through S_n element by element and $N_i(\bar{x}) = 1$ if \bar{x} at node i , and $N_i(\bar{x}) = 0$ if \bar{x} at all nodes other than i .

(d) Substituting (eq.5) into (eq.4) and solving the algebraic equations for $\hat{u}_i, (i = 1, 2, \dots, M)$.

Before performing step (d) however one has to select w_i . One possible choice is $w_i = \delta(\bar{x} - \bar{x}_i)$, where \bar{x}_i being the coordinates of arbitrary M points in Ω . This choice is termed the point-matching weighted residual method. Another choice is $w_i = N_i$, i.e., weighting functions are chosen to be exactly the coordinate functions. This leads to the Galerkin's weighted residual method.

The formulation for finite element method described above by-passes the variation principle. In the variational formulation of finite element method, a functional, I , is constructed from (eq.1). If the operator L satisfies certain conditions, then $I(u) = \frac{1}{2}(Lu, u) - (u, f)$, and u is the solution of $\delta I(u) = 0$. When the conditions do not hold, the above variational approach is invalid, however, the Galerkin's weighted residual method is still applicable in this situation.

Part B. Semi-Discrete Galerkin's Method in Spatial-Time Domain

The basic ideas discussed previously are by no means restricted to static case. It applies to both time-independent and time-dependent problems. In this part, the Galerkin's weighted residual method will be generalized to partial differential equations with time dependencies. I am going to present an explicit formulation of solving a scalar wave equation. Considering the following initial and boundary value problem:

$$\begin{aligned}
u_{tt} - c^2 \Delta u &= 0 \\
u(0, \bar{x}) &= f(\bar{x}) \\
u_t(0, \bar{x}) &= g(\bar{x}) \\
u(t, \Gamma) &= 0
\end{aligned} \tag{eq.6}$$

Here Δ is the Laplace operator, c is the phase velocity, f and g are prescribed functions. Γ is the boundary of the region Ω . Homogeneous Dirichlet boundary condition is assumed for simplicity.

One can write the approximate solution of $u(t, \bar{x})$ as

$$\hat{u}(t, \bar{x}) = \sum_{i=1}^M \hat{u}_i(t) N_i(\bar{x}) \tag{eq.7}$$

Comparing (eq.7) with (eq.5), one has noticed that N_0 is missing here because the present boundary conditions are homogeneous Dirichlet type. Besides, a_i is replaced by $\hat{u}_i(t)$, the time-dependent value of \hat{u} at node i . Here i includes all interior nodes of Ω corresponding to genuine unknowns.

Since $N_i(\bar{x}) = 0$ when \bar{x} lies on the boundary, it is guaranteed that (eq.7) satisfies boundary conditions at all time. Time dependence is reflected through the variation of \hat{u}_i with time t . This latter fact coincides with the spirit of finite element method, that is the solution is dictated by vertex values of \hat{u} . The intermediate values between the vertices result from interpolation. After finding the time dependence of these vertices values, one can obtain the time-dependence of solution everywhere in the region Ω . The separation of one variable (t here) from the remainders (\bar{x} here) and inclusion of it in the expansion coefficient as we did in (eq.7) are called semi-discrete Galerkin's method or Kantorovich method.

Now putting (eq.7) into (eq.6) leads to:

$$\hat{u}_{tt} - c^2 \Delta \hat{u} = R(t, \bar{x})$$

or

$$\sum_{i=1}^M \frac{\partial^2}{\partial t^2} \hat{u}_i(t) N_i(\bar{x}) - c^2 \sum_{i=1}^M \hat{u}_i(t) \Delta N_i(\bar{x}) = R(t, \bar{x})$$

Here $R(t, \bar{x})$ is the residual since \hat{u} is merely an approximate solution. Obviously, R is a function of t . We hope $R \approx 0$ for any $t > 0$. Employing time-independent weighting function according to the Galerkin's rule and enforcing that

$$(N_j, R(t, \bar{x})) = 0, \quad j = 1, 2, \dots, M$$

i.e.,

$$\sum_{i=1}^M \frac{\partial^2}{\partial t^2} \hat{u}_i(t) (N_j, N_i) - c^2 \sum_{i=1}^M \hat{u}_i(t) (N_j, \Delta N_i) = 0$$

If we define

$$B_{ij} = (N_i, N_j)$$

$$D_{ij} = c^2 (N_i, \Delta N_j)$$

then the above equation becomes

$$B \frac{\partial^2}{\partial t^2} \bar{u}(t) - D \bar{u}(t) = 0 \quad (\text{eq. 8})$$

Here $\bar{u}(t) = [\hat{u}_1(t), \hat{u}_2(t), \dots, \hat{u}_M(t)]^T$ with superscript T meaning transpose. Since N_j has finite support, i.e., it is non-vanishing only near the vicinity of node j, the matrices B and D are sparse.

(Eq. 8) will be supplemented by initial conditions. One has from (eq.6) and (eq.7) that

$$\sum_{i=1}^M \hat{u}_i(0) N_i(\bar{x}) = f(\bar{x})$$

$$\sum_{i=1}^M \frac{\partial}{\partial t} \hat{u}_i(0) N_i(\bar{x}) = g(\bar{x})$$

Performing inner product with N_j yields

$$\sum_{i=1}^M \hat{u}_i(0) (N_j, N_i) = (N_j, f)$$

$$\sum_{i=1}^M \frac{\partial}{\partial t} \hat{u}_i(0) (N_j, N_i) = (N_j, g)$$

Or in matrix form

$$B \bar{u}(0) = \vec{F},$$

$$B \frac{\partial}{\partial t} \bar{u}(0) = \vec{G}$$

$$\bar{u}(0) = B^{-1} \vec{F}$$

$$\frac{\partial}{\partial t} \bar{u}(0) = B^{-1} \vec{G} \quad (\text{eq.9})$$

here $\vec{F} = [F_1, F_2, \dots, F_M]^T$, $\vec{G} = [G_1, G_2, \dots, G_M]^T$, are two vectors.

$F_j = (N_j, f)$, $G_j = (N_j, g)$ stand for projections of the initial conditions f, g on the basis N_j

respectively. (Eq.8) contains M second order ordinary differential equations and (eq. 9) has 2M initial conditions. (Eq.8) and (eq.9) have well posed an initial value problem for the M variable $\hat{u}_j(t)$, $j = 1, 2, \dots, M$, the time-dependent nodal values of \hat{u} . Once $\hat{u}_j(t)$, $j = 1, 2, \dots, M$ are known, our scalar wave equation will be solved.

This method of solving wave equations is termed semi-discrete Galerkin's weighted residual method. It ends up with a set of differential equations for the time-dependent case. It is well known that in time-independent finite element method, one ends up with a set of algebraic equations for the unknowns. Algebraic equation becomes differential equation, that is the only distinction between the two cases. It is obvious that when we consider sinusoidal steady state, $\bar{u}(t) = \bar{u}^s e^{i\omega t}$, with \bar{u}^s being independent of time. Then (Eq.8) reduces to a usual eigenvalue problem, i.e.,

$$\omega^2 B \bar{u}^s + D \bar{u}^s = 0$$

in agreement to that obtained from the time-independent finite element method.

The differential equation (eq.8) can be dealt with either continuously or discretely. Assume the solution is

$$\bar{u}(t) = \bar{\alpha} e^{\lambda t} + \bar{\beta} e^{-\lambda t}$$

where $\bar{\alpha}, \bar{\beta}$ are constant vectors to be determined from the initial conditions (eq.9). Substituting the above equation into (eq.8) yields:

$$\lambda^2 B \bar{u}(t) - D \bar{u}(t) = 0$$

In order to get non-trivial solution one requires that the coefficient determinant vanishes, i.e.,

$$\det(\lambda^2 B - D) = 0$$

which yield eigenvalues λ and solutions to (eq.8).

An alternative is to write (eq.8) as difference equation in certain finite difference scheme, say, central difference with t gives:

$$B \frac{\bar{u}(n+1) - 2\bar{u}(n) + \bar{u}(n-1)}{\Delta t^2} - D \bar{u}(n) = 0$$

here $\bar{u}(n)$ stands for $t = n\Delta t$ value of $\bar{u}(t)$. The solution can be computed step by step in time direction.

At the beginning of this section we assumed Dirichlet boundary condition which is termed essential boundary condition because when we construct solutions to (eq.6) we have to choose those basis functions vanishing on the boundary. In mathematical language the admissible function space of the problem is $V = \{v(\bar{x}), v \text{ integratable}, v = 0 \text{ on the boundary}\}$. The finite element basis functions $N_i(\bar{x}), i = 1, 2, \dots, M$ consist in a subspace of V. $N_i(\bar{x})$ satisfy the

prescribed boundary conditions, i.e., $N_i(\vec{x}) = 0$ on the boundary. In the finite element method, Neumann boundary condition is called natural boundary condition because it is incorporated into weak form of the differential equation and needs not be considered in constructing solutions. In fact, it is merely guaranteed in a weighted average sense instead of being rigorously satisfied everywhere on the boundary. To be specific, let's inspect a two-dimensional example. Suppose that the initial boundary value problem is still given by (eq.6) except that the boundary condition is replaced by $\frac{\partial u}{\partial n} = q$ on Γ_1 , and $u = 0$ on Γ_0 . Here Γ_1 and Γ_0 consist in the whole boundary.

Where $\frac{\partial u}{\partial n}$ is u 's derivative along the normal direction of the boundary. Now we define admissible function space V as $V = \{v(\vec{x}), v \text{ integratable, } v = 0 \text{ on boundary } \Gamma_0\}$. Here we do not limit the value of v on Γ_1 . That gives us more room for selection of basis functions. The weak form of (eq.6) becomes

$$\int_{\Omega} u_n v dx dy - c^2 \int_{\Omega} (\Delta u) v dx dy = 0$$

employing Green's theorem to the above equation leads to

$$\int_{\Omega} u_n v dx dy - c^2 \int_{\Gamma} \frac{\partial u}{\partial n} v d\Gamma + c^2 \int_{\Omega} (u_x v_x + u_y v_y) dx dy = 0$$

where $\Gamma = \Gamma_0 + \Gamma_1$. Since $v = 0$ on Γ_0 , and $\frac{\partial u}{\partial n} = q$ on Γ_1 , we find that

$$\int_{\Omega} u_n v dx dy - c^2 \int_{\Gamma_1} q v d\Gamma_1 + c^2 \int_{\Omega} (u_x v_x + u_y v_y) dx dy = 0$$

Obviously the Neumann boundary condition has been absorbed into the above equation. In a subspace of V , we choose $N_i(x, y)$ as basis functions, with i including all node numbers except those on Γ_0 . Therefore, $N_i(x, y)$ vanishes on Γ_0 , but the Neumann boundary conditions on Γ_1 need not be satisfied, they are suppressible.

Part C. Treatment of Vector Wave Equations, Stability and Convergence

This section extends the time-dependent case to vector wave equation. In an isotropic medium, the curl Maxwell equation read:

$$\begin{aligned} \vec{\nabla} \times \vec{E} &= -\mu \frac{\partial \vec{H}}{\partial t} \\ \vec{\nabla} \times \vec{H} &= \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t} \end{aligned} \tag{eq. 10}$$

Where μ, σ, ε are magnetic permeability, electric conductivity, and dielectric permittivity respectively. The dielectric permittivity and magnetic permeability of the medium are allowed to be functions of position as well as electric and magnetic fields, respectively, in the nonlinear case. (Eq.10) constitutes a hyperbolic system which must satisfy specific initial and boundary conditions, i.e., $\vec{E}(\vec{x},0), \vec{H}(\vec{x},0)$ are given everywhere in the computational domain. Tangential components of \vec{E} or \vec{H} on the boundary must be known for all $t \geq 0$. For the boundary at the infinity, Sommerfeld's radiation condition must hold, that is the scattered fields must be of an outgoing traveling wave type.

Assume \vec{E} and \vec{H} have the form

$$\begin{aligned}\vec{E} &= \sum_{i=1}^M \phi_i(\vec{x}) \vec{E}_i(t) \\ \vec{H} &= \sum_{j=1}^N \psi_j(\vec{x}) \vec{H}_j(t)\end{aligned}\tag{eq. 11}$$

In contrast to time-domain finite difference method, this scheme positions all the components of electric field on the same node, the same is true for the magnetic field. The basis functions ϕ_i and ψ_j interpolate the fields within each element using the values on the nodes forming the element. Functions ϕ_i and ψ_j are chosen to satisfy given boundary conditions. Substitution of (eq. 11) into (eq. 10) leads to

$$\begin{aligned}\sum_{i=1}^M (\psi_l, \vec{\nabla} \phi_i(\vec{x})) \times \vec{E}_i(t) &= -\mu \sum_{j=1}^N (\psi_l, \psi_j) \frac{\partial \vec{H}_j(t)}{\partial t} \\ \sum_{j=1}^N (\phi_k, \vec{\nabla} \psi_j(\vec{x})) \times \vec{H}_j(t) &= \sum_{i=1}^M (\phi_k, \phi_i) [\sigma \vec{E}_i(t) + \varepsilon \frac{\partial \vec{E}_i(t)}{\partial t}]\end{aligned}$$

If we define

$$\begin{aligned}\vec{P}_{li} &= (\psi_l, \vec{\nabla} \phi_i) \\ \vec{Q}_{kj} &= (\phi_k, \vec{\nabla} \psi_j) \\ B_{lj} &= (\psi_l, \psi_j) \\ D_{ki} &= (\phi_k, \phi_i)\end{aligned}$$

Then the above equations can be written as

$$\sum_{i=1}^M \vec{P}_{li} \times \vec{E}_i(t) = -\mu \sum_{j=1}^N B_{lj} \frac{\partial \vec{H}_j(t)}{\partial t}\tag{eq.12}$$

$$\sum_{j=1}^N \vec{Q}_{kj} \times \vec{H}_j(t) = \sum_{i=1}^M D_{ki} [\sigma \vec{E}_i + \varepsilon \frac{\partial \vec{E}_i(t)}{\partial t}] \quad (\text{eq. 13})$$

(Eq.12) and (eq.13) are coupled differential equations for $\vec{E}_i(t)$ and $\vec{H}_j(t)$, the nodal values of electric and magnetic field respectively, which are unknowns of our problem. The coupled differential equations may be solved by finite difference method, for instance, the leap-frog scheme. Although the use of a single mesh seems attractive, it turns out that for the simplicity of computation it is preferable to distribute the electric and magnetic field nodes in such a way that they are mutually interleaved. The electric and magnetic fields are defined on two complementary grids so that the nodes of the former are positioned inside the elements of the latter. Eventually, the equations are discretized as:

$$\sum_{i=1}^M \vec{P}_{li} \times \vec{E}_i^n = -\mu \sum_{j=1}^N B_{lj} \frac{\vec{H}_j^{n+1/2} - \vec{H}_j^{n-1/2}}{\Delta t} \quad (\text{eq. 14})$$

$$\sum_{j=1}^N \vec{Q}_{kj} \times \vec{H}_j^{n+1/2} = \sum_{i=1}^M D_{ki} [\sigma \frac{\vec{E}_i^n + \vec{E}_i^{n+1}}{2} + \varepsilon \frac{\vec{E}_i^{n+1} - \vec{E}_i^n}{\Delta t}] \quad (\text{eq. 15})$$

Where

$$\begin{aligned} \vec{E}_i^n &= \vec{E}_i(t = n\Delta t) \\ \vec{H}_j^{n+1/2} &= \vec{H}_j(t = (n+1/2)\Delta t) \end{aligned}$$

Once the initial conditions \vec{E}_i^0 and $\vec{H}_j^{-1/2}$ are known, \vec{E}_i^n and $\vec{H}_j^{n+1/2}$ can be updated and then (eq. 11) leads to approximate solutions to electric and magnetic fields everywhere at any time instant provided that matrices $\vec{P}_{li}, \vec{Q}_{kj}, B_{lj}, D_{ki}$ are already assembled from element-by-element integration.

Finally we scratch the surface of convergence and stability in conjunction with discretization.

The problem of convergence means that the difference between the theoretical solutions of the differential equation and the discretized equation at a fixed point (\vec{x}, t) tends to zero uniformly, as the mesh size is refined in such a way that $h, \Delta t$ approach zero while m_i, n approach infinity, with $m_i h = x_i$ and $n\Delta t = t$ both remaining fixed. In the above, $i = 1, 2, 3$.

The problem of stability means that the difference between the theoretical solution and numerical solution of the discretized equation remains bounded as n tends to infinity while $n\Delta t = t$ is fixed.

It has been shown that if a linear difference equation is consistent with a properly posed linear initial value problem, then stability is the necessary and sufficient condition for convergence.

The hyperbolic system (eq. 10) for $t \in [0, \infty]$ with initial conditions

$\vec{E}(\vec{x}, 0) = \vec{F}_1(\vec{x}), \vec{H}(\vec{x}, 0) = \vec{F}_2(\vec{x})$ on a domain without boundaries is a properly posed initial value problem termed a Cauchy problem. Since the leap-frog finite difference scheme introduced above is consistent with the Cauchy problem, we merely need to examine the conditions for stability. Concerning stability, it has been shown that for regular square grid of mesh size h , the leap-frog scheme is stable if $c\Delta t \leq h$, here c is the phase velocity of propagation. It is interesting to notice that the stability criterion is independent of spatial dimensions if the computational grid is uniform, i.e., the mesh size increment h is the same along any directions such as x, y , and z . While in time-domain finite difference method, the stability condition depends on dimension d in the manner of $c\Delta t \leq h/\sqrt{d}$. This is an advantage of time domain finite element method over time domain finite difference method in higher spatial dimensions.

Part D. Concluding Remarks

The finite element method is recognized as integration method. In present case however, not all variables (\vec{x}, t) are handled in the integration manner, since t is treated as differentially or discretely. This is why the method got its name “semi-discrete method”, The semi-discrete Galerkin’s weighted residual method combines merits of finite difference and finite element methods. In time domain, it regards t as discrete variable and possesses the simplicity of finite difference method; while in spatial domain, it duplicates time-independent finite element method and inherits its advantages. It is expected that this time-domain finite element method can well deal with complex geometries, Neumann boundary conditions and inhomogeneous materials in computational electromagnetics.

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